Optimal verification of LTI discrete-time systems

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Outline

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Verification

- **Numerical methods**
  The discretization of $\ddot{x} + \dot{x} + x = 0$ by a Euler scheme with initial conditions in $[0,1]^2$ (position,speed). Let $h = 0.01$.

$$\begin{pmatrix} x_{k+1} \\ v_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & h \\ -h & 1-h \end{pmatrix} \begin{pmatrix} x_k \\ v_k \end{pmatrix}$$

- **Programs**

```
x = [1,2];
y = [1,2];
while (x^2 + v^2>=1) {
  ox = x;
  oy = y;
  x = 0.5*ox-0.4*oy
  y = ox-0.5*oy;
 }
```

Properties to prove

Some interesting properties on the examples :

1. The values are bounded? The output values are both smaller than 1?
2. Can we leave the loop for all possible initial values? Number of iterations?

Other interesting properties in general :
Robustness, termination, reachability...

The problem formulation

Inputs

- Linear System with d states

$$x_0 \in X^{\mathrm{in}},\ x_{k+1} = A x_k,\ k \in \mathbb{N};$$

  where X^{in} is a polytope.

- Property of the form:

$$\forall\, k \in \mathbb{N},\ x_k \in \{y \in \mathbb{R}^d \mid y^{\intercal} Q y \ll \alpha\}$$

  where Q is symmetric and $\alpha \in \mathbb{R} \cup \{+\infty\}$.

Output

A proof of the property or a counterexample.

On the examples

- **First example**
  - Boundedness:
    $$\forall\, k \in \mathbb{N},\ (x_k, v_k)\,\mathrm{Id}(x_k, v_k)^\intercal < +\infty$$
    And we want the maximal Euclidean norm
  - Output values ≤ 1?
    $$\forall\, k \in \mathbb{N},\ (x_k, v_k)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}(x_k, v_k)^\intercal \leq 1$$
    and
    $$\forall\, k \in \mathbb{N},\ (x_k, v_k)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}(x_k, v_k)^\intercal \leq 1$$
- **Second example** :
  Not formulated as a sublevel set but the proposed method will solve the problem

Outline

Formulation

To prove the property:

$$\forall\, k \in \mathbb{N},\ x_k^\intercal Q x_k \ll \alpha$$

can be reduced to prove:

$$\sup_{k \in \mathbb{N}} \sup_{x \in X^{\mathrm{in}}} x^\intercal {A^k}^\intercal Q A^k x \ll \alpha$$

To prove or disprove the property it suffices to compute :

$$\sup_{k \in \mathbb{N}} \sup_{x \in X^{\mathrm{in}}} x^\intercal {A^k}^\intercal Q A^k x$$

Infinities

1. The function
$$f : x \mapsto \sup_{k\in\mathbb{N}} x^\intercal A^{k\intercal} Q A^k x$$
is not quadratic nor polynomial nor convex/concave a priori.
Thus
$$\sup_{x\in X^{\mathrm{in}}} f(x)$$
cannot be solved exactly and an overapproximation cannot be computed easily.
2. The evaluation of f requires an infinite number of computations but we can use :
$$f_k : x \mapsto x^\intercal A^{k\intercal} Q A^k x.$$
However, for all $k \in \mathbb{N}$, we have to solve a NP-Hard problem.

Then to solve exactly the problem we have to finitely discretized the problem:

- for X^{in};
- for $k \in \mathbb{N}$.

Outline

Using convexity

We recall the well-known lemma:

Lemma (Supremum of convex functions over compact convex sets)

Let $g : \mathbb{R}^d \mapsto \mathbb{R}$ be a convex and D be a convex compact set. Then :

$$\sup_{x \in D} g(x) = \sup_{x \in \mathcal{E}(D)} g(x)$$

where $\mathcal{E}(D)$ denotes the set of extreme points of D.

If $Q \succeq 0$, then$\forall\, k \in \mathbb{N}$, $f_k : x \mapsto x^\intercal A^{k\intercal} Q A^k x$ is convex.

- X^{in} is a polytope then $\mathcal{E}(X^{\mathrm{in}})$ is a finite set.
- We compute once $\mathcal{E}(X^{\mathrm{in}})$.

Discretisation of the infinite sequence

Now assume $Q \succeq 0$. Then for all $k \in \mathbb{N}$, $\sup_{x \in X^{\mathrm{in}}} x^{\intercal} {A^k}^{\intercal} Q A^k x$ can be computed exactly in finite time .

The problem is to compute K such that:

$$\sup_{k \in \mathbb{N}} \sup_{x \in X^{\mathrm{in}}} x^{\intercal} {A^k}^{\intercal} Q A^k x = \sup_{k \in [K]} \sup_{x \in X^{\mathrm{in}}} x^{\intercal} {A^k}^{\intercal} Q A^k x$$

Discussion about hypotheses

Since $Q \succeq 0$, we should ask for the boundedness of $\{A^k x, k \in \mathbb{N}\}$, $\forall x \in X^{\mathrm{in}}$.

Indeed:

- If $Q \succ 0$, then Q induces a norm on $\mathbb{R}^d$. Hence

$$x^\intercal A^{k\intercal} Q A^k x = \|A^k x\|_Q < +\infty \iff (A^k x)_{k\in\mathbb{N}} \text{ bounded.}$$

- If $Q \succeq 0$, we can allow unbounded sequences in the null space of Q.

The boundedness allows $\rho(A) = 1$.

To simplify the problem, we assume for $A^k \mapsto 0 \iff \rho(A) < 1$.

Finally we assume $Q \succeq 0$ and $\rho(A) < 1$.

Theorem (Computable integer)

There exists a computable K such that for all $x \in X^{\mathrm{in}}$,

$$\sup_{k\in\mathbb{N}} x^\intercal A^{k\intercal} Q A^k x = \sup_{k\in[K]} x^\intercal A^{k\intercal} Q A^k x$$

Matrix norms

Matrix norms :

- Norms on $\mathbb{R}^{d\times d}$ (sub-additive and strictly positive);
- Sub-multiplicative :$N(AB) \leq N(A)N(B)$.

The sub-multiplicative property implies $N(A^k) \leq N(A)^k$.

For every norm $\|\cdot\|$ over $\mathbb{R}^d$, the map

$$N(A) = \sup_{x\in\mathbb{R}^d\setminus\{0\}} \frac{\|Ax\|}{\|x\|}$$

is a matrix norm.

Rayleigh quotient

Let $B \succeq 0$ and $C \succ 0$.
Raleigh quotient is defined, for all $x \in \mathbb{R}^d \backslash \{0\}$ by

$$\frac{x^\intercal Bx}{x^\intercal Cx}$$

Two quantities are interesting:

$$\begin{cases} \text{sup of Raleigh quotient} = \lambda_{\max}(C^{-1/2}BC^{-1/2}) \\ \text{inf of Raleigh quotient} = \lambda_{\min}(C^{-1/2}BC^{-1/2}) \end{cases}$$

Special case : $C = \text{Id}$, the sup is $\lambda_{\max}(B)$ and the inf is $\lambda_{\min}(B)$.

A first idea

Let $x \in X^{\text{in}}$.

$$\begin{aligned} x^\intercal {A^k}^\intercal Q A^k x &\leq \lambda_{\max}(Q)\|A^k x\|_2^2 && \text{From Rayleigh quotient} \\ &\leq \lambda_{\max}(Q)\|A^k\|_2^2\|x\|_2^2 && \text{From norm operator def.} \\ &\leq \lambda_{\max}(Q)\|A\|_2^{2k}\|x\|_2^2 && \text{From matrix norm def.} \end{aligned}$$

Now let define, for $B \succeq 0$, $\mu(B) = \sup_{x \in X^{\text{in}}} x^\intercal B x$.

We impose for K that $x^\intercal {A^k}^\intercal Q A^k x \leq \sup_{x \in X^{\text{in}}} x^\intercal Q x = \mu(Q)$ for all $k \geq K$.

We have to exhibit a lower bound on integers k:

$$\|A\|_2^{2k} \leq \mu(Q)\lambda_{\max}(Q)^{-1}\mu(Id)^{-1}$$

Using ln, if $\|A\|_2^2 < 1$ we get :

$$k \geq \frac{\ln(\mu(Q)\lambda_{\max}(Q)^{-1}\mu(Id)^{-1})}{\ln(\|A\|_2^2)}$$

Needs a Lyapunov function

Two remarks:

- The assumption $\|A\|_2 < 1$ is very restrictive.
- We have to check whether

$$\ln(\mu(Q)\lambda_{\max}(Q)^{-1}\mu(Id)^{-1}) \geq 0 \iff \mu(Q)\lambda_{\max}(Q)\mu(Id) \leq 1$$

Needs a Lyapunov function

The condition $\rho(A) < 1$ the existence of a matrix norm $\|\cdot\|$ such that $\|A\| < 1$. Let P such that $P \succ 0$ and $P - A^\intercal PA \succ 0$ (exists since $\rho(A) < 1$). Then $x \mapsto \sqrt{x^\intercal Px}$ is a norm over $\mathbb{R}^d$ and then $\|A\|_P^2 = \sup_{x\in\mathbb{R}^d\setminus\{0\}} \dfrac{x^\intercal A^\intercal PAx}{x^\intercal Px}$ is a matrix norm.

Proposition

$0 < \|A\|_P < 1$ *and* $\mu(Q)\mu(P)^{-1}\lambda_{\max}(Q)\lambda_{\min}(P)^{-1}$.

Proof.

From Weyl's inequalities i.e. for symmetric matrices M, N :

$$\lambda_k(M) + \lambda_{\min}(N) \leq \lambda_k(M+N) \leq \lambda_k(M) + \lambda_{\max}(N)$$

and $\rho(A) \leq \|A\|_P$. □

Final integers

Let P a solution of the discrete Lyapunov equation:

$$K = E\left[\frac{\ln(\mu(Q)\mu(P)^{-1}\lambda_{\max}(Q)\lambda_{\min}(P)^{-1})}{\ln(\|A\|_P^2)}\right] + 1$$

Final integers

Let P a solution of the discrete Lyapunov equation:

$$K = E\left[\frac{\ln(\mu(Q)\mu(P)^{-1}\lambda_{\max}(Q)\lambda_{\min}(P)^{-1})}{\ln(\|A\|_P^2)}\right] + 1$$

or using Rayleigh quotient:

$$K_1 = E\left[\frac{\ln(\mu(Q)\mu(P)^{-1}\lambda_{\max}(P^{-1/2}QP^{-1/2}))}{\ln(\|A\|_P^2)}\right] + 1$$

We can define for a given $t > 0$, a matrix P such that $tP \succeq Q$ and $P - A^\intercal PA \succ 0$:

$$K_t = E\left[\frac{\ln(\mu(Q)\mu(P)^{-1}t^{-1})}{\ln(\|A\|_P^2)}\right] + 1$$

Outline

Example - Discretisation of Harmonic Oscillator

Let recall the first example:

$$\begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \in [0,1]^2, \begin{pmatrix} x_{k+1} \\ v_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & h \\ -h & 1-h \end{pmatrix} \begin{pmatrix} x_k \\ v_k \end{pmatrix}$$

We choose $t = \lambda_{\max}(Q)$ for K_t.

- **Boundedness**:

$$\text{For } Q = \text{Id}, K = 169, K_1 = 169, K_t = 133.$$

$$\text{Max } \|(x_k, v_k)\|_2 = 2 \text{ at } k = 0 \text{ for } (x_0, v_0) = (1,1);$$

- **Maximal value of x_k?**

$$\text{For } Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, K = 296, K_1 = 188, K_t = 230.$$

Max=1.6489 at $k = 61$ for vector =(1,1);The property $x_k \leq 1$ is false.

- **Maximal value of v_k?**

$$\text{For } Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, K = 296, K_1 = 261, K_t = 228.$$

Max=1 at $k = 0$ for vector =(1,1);

Example - Leaving the loop

In the second example the data are

$$X^{\mathrm{in}} = [1,2]^2,\ A = \begin{pmatrix} 0.5 & -0.4 \\ 1 & -0.5 \end{pmatrix} \text{ and } Q = \mathrm{Id}\,.$$

To apply the previous method we have to replace $\mu(Q)$ by 1 in K, K_1 and K_t.

Indeed, we constructed K such that $\sup_{x \in X^{\mathrm{in}}} x^\intercal A^{k\intercal} Q A^k x \leq \mu(Q)$ for all $k \geq K$.

Here we are interested in the first $\bar{k}$ such that $\sup_{x \in X^{\mathrm{in}}} x^\intercal A^{\bar{k}\intercal} Q A^{\bar{k}} x \leq 1$.

For example, we compute

$$K = E\left[\frac{\ln(\mu(P)^{-1}\lambda_{\max}(Q)\lambda_{\min}(P)^{-1})}{\ln(\|A\|_P^2)}\right] + 1 = 11$$

The modified $K_1 = 11$ and the modified $K_t = 281$.

We can take the smallest integer thus 11.

Experiments

Let us test the Matlab code.

Outline

Optimize the integers

The integers K, K_1, K_t can be big.

We should solve a minimization problem where P the a decision variable.

Start by solving the problem

$$\mathrm{Min}\{\|A\|_P \mid P \succ 0\}$$

+ The function $P \mapsto \|A\|_P$ is quasi-convex;
+ Bounded from below by $\rho(A)$;
− The constraints set is not closed.

About affine and non-linear dynamics

- Affine case :
  - Lift-and-Project (allow 1 as eigenvalue);
  - Use the closed form for $x_k = A^k x_0 + \sum_{j=0}^{k-1} A^j b$.

- Non-linear dynamics :
  - Norm operator Lyapunov functions ;
  - Non-linear spectral radii (warning well-defined on pointed cones) or joint spectral radii.

Conclusion

- Succeed to solve exactly optimization problems over reachable values constraints set with a finite number of evaluations.
- Succeed to compute global stopping criteria for stable linear systems and ellipsoidal properties.